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Arithmetic Subgroups and Applications

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Abstract

Arithmetic subgroups are an important source of discrete groups acting freely on manifolds. We need to know that there exist many torsion-free ??(?,?) is an ?arithmetic? subgroup of ??(?,?). The other arithmetic subgroups are not as obvious, but they can be constructed by using quaternion algebras. Replacing the quaternion algebras with larger division algebras yields many arithmetic subgroups of ??(?,?), with ??>2. In fact, a calculation of group cohomology shows that the only other way to construct arithmetic subgroups of ??(?,?) is by using arithmetic groups. In this paper justifies Commensurable groups, and some definitions and examples,?-forms of classical simple groups over ?, calculating the complexification of each classical group, Applications to manifolds. Let us start with ??(??,?). This is already a complex Lie group, but we can think of it as a real Lie group of twice the dimension. As such, it has a complexification.

Index terms — lie group, commensurable groups, orthogonal group, symplectic group, subgroups.

1 I. Introduction

In This paper we will give a quite explicit description of the arithmetic subgroups of almost every classical Lie group ??. (Recall that a simple Lie group ??is "classical" if it is eithera special linear group, an orthogonal group, a unitary group, or a symplectic group The key point is that all the ?-forms of ??arealso classical, not exceptional, so they are fairly easy to understand. However, there is an exception to this rule, some 8-dimensional orthogonal groups have ?-forms of so-called triality type, that are not classical and will not bediscussed in any detail here given ??, which is a Lie group over ?, we would like to know all of its?-forms (because, by definition, arithmetic groups are made from ?-forms) [1,2,3]. However, we will start with the somewhat simpler problem that replaces the fields ?and ?with the fields ?and ?: finding the ?-forms of the classicalLie groups over ?.In this paper we construct methods by arithmetic groups. The associated symmetric space ???? ?? (?) = ???? ?? is the hyperbolic plane ? ?? . There are uncountably many lattices in ???? ?? (?) (with theassociated locally symmetric spaces being nothing other than Riemann surfaces), but only countably many of them are arithmetic. But in higher rank Lie groups, there is the following truly remarkable theorem known as Margulis arithmeticity. Let ?? be a connected semi simple Lie group with trivial centre and no compact factors, and assume that the real rank of ?? is at least two. Then every irreducible lattice ?? ? ?? is arithmetic [4,5, **??**]. These groups play a fundamental role in number theory, and especially in the study of automorphic forms, which can be viewed as complex valued functions on a symmetric domain which are invariant under the action ofan arithmetic group. Appeared that some arithmetic groups are the symmetry groups of several string theories. This is probably why this survey fits into these proceedings [7].

2 II. Commensurable Groups

Subgroups ?? 1 and ?? 2 of a group are said to be commensurable if ?? [8,9].

3 a) Definition

Let ?? 1 and ?? 2 be subgroups of a group ??. We say that ?? 1 and **??**6,10] b) Remark "Being commensurable" is an equivalence relation [11].?? 2 are commensurable if [?? 1 : ??], [?? 2 : ??] < ?, where ?? = ?? 1 ? ?? 2 ,

Examples ?? finite: any two ?? 1 and ?? 2 are commensurable. ?? = ??: ?? 1 and ?? 2 are commensurable iff they are isomorphic [13].

4 d) The geometry topology

Let ?? 1 and ?? 2 as fundamental groups.For instance, let ?? = ??????(??, ?) acting on ?? 3 and let ?? 1 and ?? 2 be lattices which are fundamental groups of hyperbolic manifolds (or, more generally, or bifolds). if ?? 1 and ?? 2 are commensurable ??/?? ?? and ?? ?? ?? have a common finite cover. Since (orbifold) fundamental groups are defined as subgroups of ?? only up to conjugacy, it is natural to allow subgroups to have a finite index intersection only up to conjugacy [5,4].

5 e) Definition

Let ?? 1 and ?? 2 be subgroups of a group G. We say that ?? 1 and ?? 2 are weakly commensurable if there is a g in ?? suchthat [?? 1 : ??], [?? 2 : ??] < ?, where ?? = ?? 1 ? ???? 2 ?? ??? [9] Remark "Weak commensurability" is also an equivalence relation [13].

6 Ref g) The Geometry Topology in dimension 2

Let ?? ð ??"ð ??" denote the fundamental group of the genus ð ??"ð ??" close orientable surface. of course ?? ð ??"ð ??" ? ?? 2 for all ð ??"ð ??" ? 2. On the other hand one can find discrete surface groups ?? 1 and ?? 2 inside ?? = ??????(2, ??) which are not (weakly) commensurable [14].

7 h) Definition

Let ?? 1 and ?? 2 be groups. We say that ?? 1 and ?? 2 are abstractly commensurable if there are subgroups ?? ?? of ?? ?? ,?? = 1,2, such that [?? 1 : ?? 1], [?? 2 : ?? 2] < ? and ?? 1 ? ?? 2 [5].

8 III. Definitions

Let ?? be an algebraic group over ?. Let ??: ?? ? ???? ?? be a faithful representation of ??ona finite-dimensional vector space ??, and let ??be a lattice in ??.Define .??(?) ?? = {ð ??"ð ??" ? ??(?)|??(ð ??"ð ??")?? = ??}.**(1)**

Anarithmetic subgroup of ??(?) is any subgroup commensurable with ??(?) ?? . For an integer ?? > 1, the principal congruence subgroup of level N is:Î?"(??) ?? = {ð ??"ð ??" ? ??(?) ?? |ð ??"ð ??" ???????? ???? 1 ???? ??/????}**(2)**

In other words,.Î?"(??) ?? is the kernel of??(?) ?? ? ?????? (??/????).

In particular, it is normal and of finite index in ??(?) ?? congruence subgroup of ??(?) is any subgroup containing some Î?"(??) as a subgroup of finite index, so congruence subgroups are arithmetic subgroups [15].

9 a) Example

Let ?? = ???? ?? with its standard representation on ? ?? and its standard lattice ?? = ?? ?? . Then ??(?) ?? consists of the ?? = ???? ?? (?) such that ??? ?? = ? ?? . On applying ?? to ?? 1 , ? , ?? ?? , we see that this implies that ?? has entries in ?. Since ?? ?1 ? ?? = ? ?? , the same is true of ?? ?1 . Therefore, ??(?) ?? is:???? ?? (?) = {?? ? ?? ?? (?)|??????(??) = ±1}**(3)**

The arithmetic subgroups of ???? ?? (?) are those commensurable with ???? ?? (?) [16]. By definition,Î?"(??) = {?? ? ???? ?? (?)| ?? ? ?? ?????? ??} = ???? ???? ? ? ???? ?? (?)|?? ?????????????? ??? ???? ? ?? ???? ??,**(4)**

Which is the kernel of???? ?? (?) ? ???? ?? (?/???).

10 b) Example

The group Ref is an arithmetic subgroup of ???? 2?? (?), and all arithmetic subgroups are commensurable with it [3].???? 2?? (?) = ??? ? ???? 2?? (?)| ?? ?? ? 0 ?? ??? 0 ? ?? = ? 0 ?? ??? 0 ??**(5)**

11 IV. R-Forms of Classical Simple Groups Over ?

To set the stage, let us recall the classical result that almost all complex simple groups are classical: a) Theorem All but finitely many of the simple Lie groups over ?are isogenous to either ????(??, ?), ????(??, ?), or ????(2??, ??), for some ??, [15,17] b) Remark Up to isogeny, there are exactly five simple Lie groups over ?that are not classical. They are the "exceptional" simple groups, and are called ?? ð ??"ð ??" , ?? ?? , ?? ?? , ?? 4 , and ?? 2 .

We would like to describe the ?-forms of each of the classical groups. For example, finding all the ?-forms of ????(??, ?) would mean making a list of the (simple) Lie groups G, such that the "complexification" of ?? is ????(??, ?). This is not difficult, but we should perhaps begin by explaining more clearly what it means. It has already been mentioned that, intuitively, the complexification of ??is the complex Lie group that is obtained from ??by replacing real numbers with complex numbers. For example, the complexification of ????(??, ?) is ????(??, ?). In general, ??is (isogenous to) the set of real solutions of a certain set of equations, and we let ?? ? be the set of complex solutions of the same set of equations [18] c) Notation of complex, semisimple Assume ?? ? ????(â??", ?), for some â??". Since ??is almost Zariski closed, there is a certain subset ??of?[?? 1,1 , ? , ?? â??",â??"], such that ?? ? = ??????(??) ? . Let:?? ? = ?????? ? (??) = { ð ??"ð ??" ? ????(â??", ?)|??(ð ??"ð ??") = 0, for all ?? ? ??}.**(6)**

Then ?? ? is a (complex, semisimple) Lie group.

12 d) Example

? ????(??, ?) ? = ????(??, ?).

? ????(??) ? = ????(??, ?).

? ????(??, ??) ? ? ????(?? + ??, ?).

13 e) Definition

If ?? ? is isomorphic to ??, then we say that

? ??is the complexification of ??, and that

? ??is an ?-form of ?? The following result lists the complexification of each classical group. It is not difficult to memorize the correspondence. For example, it is obvious from the notation that the complexification of ????(??, ??) should be symplectic.Indeed, the only case that really requires memorization is the complexificationof ????(??, ??) **??**19,2].

14 Ref f) Proposition

Here is the complexification of each classical Lie group.

Real forms of special linear group:1 ????(??, ?) ? = ????(??, ?), 2 ????(??, ?) ? ? ????(??, ?) × ????(??, ?), 3 ????(??, ?) ? ? ????(2??, ?), 4 ????(??, ??) ? ? ????(?? + ??, ?).

Real forms of orthogonal groups:1 ????(??, ??) ? ? ????(?? + ??, ?), 2 ????(??, ?) ? ? ????(??, ?) × ????(??, ?), 3 ????(??, ?) ? ? ????(2??, ?).

Real forms of symplectic groups:1 ????(??, ?) ? = ????(??, ?), 2 ????(??, ?) ? ? ????(??, ?) × ????(??, ?), 3 ????(??, ??) ? ? ????(2(?? + ??), ?). V.

Calculating the Complexification of Classical G

Here is justifies Proposition 4.6, by calculating the complexification of each classical group. Let us start with ????(??, ?). This is already a complex Lie group, but wecan think of it as a real Lie group of twice the dimension. As such, it has a complexification **??**20, **??**].

15 a) Lemma

The tensor product ??? ? ? is isomorphic to ?????? 2×2 (?).

Proof. Define an ?-linear map ??: ? ? ?????? 2×2 (?) by??(1) = ????, ??(??) = ? ?? 0 0 ??? ? , ??(??) = ? 0 1 ?1 0 ? , ??(??) = ? 0 ?? ?? 0 ?**(7)**

It is straight forward to verify that ??is an injective ring homomorphism. Furthermore, ??({1, ??, ??, ??}) is a ? ? basis of ?????? 2×2 (?). Therefore, the map ?? ? : ? ? ? ? ?????? 2×2 (?) defined by ?? ? (?? ? ??) = ??(??) ??is a ring isomorphism [1] b) How to find the real forms of complex groups in ? Now, we will explain how to find all of the possible ? ? forms of ????(??, ?). We take an algebraic approach, based on Galois theory, and we first review the most basic terminology from the theory of (nonabelian) group cohomology [1].

16 c) Definitions

Suppose a group ??acts (on the left) by automorphismson a group ??. (For ?? ? ?? and ?? ? ??, we write ?? ?? for the image of munder ??.) A function ??: ?? ? ?? is a 1-cocycle (or "crossed homomorphism") if ??(????) = ??(??) ? ?? ?? (??)for all??; ?? ? ??.

Two 1-cocycles ?? and ?? are equivalent (or "cohomologous") if there is some ?? ? ??, such that

17 d) Galoiscohomology

For convenience, let ?? ? = ????(??, ?). Suppose ??: ?? ? ? ????(??, ?) is an embedding, such that ??(?? ?)is defined over ?. We wish to find all the possibilities for the group ??(?? ?) ? = ??(?? ?) ? ????(??, ?) that can be obtained by considering all the possible choices of??.Let ??denote complex conjugation, the nontrivial Galois automorphism of C over R. Since ? = {?? ? ?|?(??) = ??}, we have????(??, ?) = {ð ??"ð ??" ? ????(??, ?)|?(ð ??"ð ??") = ð ??"ð ??"},**(8)**

where we apply ?? to a matrix by applying it to each of the matrix entries.Therefore??(?? ?) ? = ??(?? ?) ? ????(??, ?) = {ð ??"ð ??" ? ??(?? ?)|?(ð ??"ð ??") = ð ??"ð ??"}**(9)**

Since ??(?? ?) is defined over ?, we know that it is invariant under ??, so wehave?? ? ?? ? ??(?? ?) ?? ? ??(?? ?) ?? ?1 ? ? ? ?? ? .

Let ?? ? = ?? ?1 ????: ?? ? ? ?? ? be the composition. Then the real form corresponding to ?? is?? ? = ?? ?1 ???(?? ?) ? ????(??, ?)? = {ð ??"ð ??" ? ?? ? |?? ?(ð ??"ð ??") = ð ??"ð ??"}**(10)**

To summarize, the obvious ?-form of ?? ? is the set of fixed points of the usual complex conjugation, and any other ?-form is the set of fixed points of some other auto morphism of ?? ? . Now let??(??) = ?? ??? ?1 : ?? ? ? ?? ? .**(11)**

It is not difficult to see that ? ??(??) is an automorphism of ?? ? (as an abstract group), and

? ??(??) is holomorphic (since ?? ?1 and ?????? ?1 are holomorphic -in fact, they can be represented by polynomials in local coordinates). So ??(??) ? Aut(?? ?). Thus, by defining ??(1) to be the trivial automorphism,we obtain a function ??: Gal(?/?) ? Aut(?? ?). Let Gal(?/?) act on Aut(?? ?), by defining?? ?? = ?????? ?1 for?? ? Aut(?? ?).**(12)**

Then ??(??) = ?? ?1 ?? ?? , ???? ??(??) ? ?? ?? (??) = ??(1) (since ?? 2 = 1). This means that??is 1 ? ?????????????? of group cohomology, and therefore defines an element of the cohomology set ? 1 Gal(?/?), Aut(?? ?). In fact: This construction provides a one-to-one correspondence between ? 1 Gal(?/?), Aut(?? ?). and the set of ? -forms of ?? ? [21, 14, **??**9].

18 VI.

Applications to Manifolds ? ?? ?? is a discrete subset of ? ?? ?? , i.e., every point of ? ?? ?? has an open neighbourhood(for the real topology) containing no other point of ? ?? ?? , Therefore, ???? ?? (?) is discrete in ???? ?? (?) and it follows that every arithmetic subgroup Î?" of a group ??is discrete in ??(?)Let ??be an algebraic group over ?. Then ??(?)is a Lie group, and for every compactsubgroup ??of ??(?), ?? = ??(?)/??is a smooth manifold **??**22]. a) Torsion-free arithmetic groups ???? 2 (?) is not torsion-free. For example, the following elements have finite order:? ?1 0 0 ?1 ? 2 = ? 1 0 0 1 ?, ? 0 ?1 1 0 ? 2 = ? ?1 0 0 ?1 ? = ? 0 ?1 1 1 ? 3**(13)**

b) Theorem

Every arithmetic group contains a torsion-free subgroup of finite index.For this, it suffices to prove the following Lemma [12].

19 c) Lemma

For any prime ?? ? 3, the subgroup ??(??)/ of ???? ?? (?) is torsion-free.

Proof. If not, it will contain an element of order a prime â??" and so we will have an equation:(1 + ?? ?? ??) ð ?"µð ?"µ = ??**(14)**

with ?? ? 1and ??a matrix in ?? ?? (?) not divisible by ??. Since ?? and ?? commute, we can expand this using the binomial theorem, and obtain an equation:â??"?? ?? ?? = ? ? ? â??" ?? ? â??" ??=?? ?? ???? ?? ??**(15)**

In the case that â??" ? ??, the exact power of ??dividing the left hand side is ?? ?? , but ?? 2?? divides the right hand side, and so we have a contradiction. In the case that â??" = ??, the exact power of ??dividing the left hand side is ?? ?? +1 , but, for 2 ? ?? < ??, ?? 2?? +1 | ? ?? ?? ? ?? ???? because ?? ? ?? ?? ?, and ?? 2?? +1 |?? ???? because ?? ? 3. A gain we have a contradiction [12]. d) Application to quadratic forms Consider a binary quadratic form:??(??, ??) = ???? 2 + ?????? + ???? 2 , ??, ??, ?? ? ? (**16**)

Assume ?? is positive definite, so that its discriminant ?= ?? 2 ? 4???? < 0. There are many questions one can ask about such forms. For example, for which integers ?? is there a solution to ??(??, ??) = ?? with ??, ?? ? ? For this, and other questions,the answer depends only on the equivalence class of ??, where two forms are said to be equivalent if each can be obtained from the other by an integer change of variables. More precisely, ??and ?? ? are equivalent if there is a matrix ?? ? ???? 2 (?) taking ?? into ?? ? by thechange of variables, Ref? ??? ??? ? = ?? ? ?? ?? ?.**(17)**

In other words, the forms:??(??, ??) = (??, ??). ??. ? ?? ?? ?, ?? ? (??, ??) = (??, ??) . ?? ? . ? ?? ?? ?**(18)**

are equivalent if ?? = ?? ?? . ?? ? . ?? for ?? ? ???? 2 (?). Every positive-definite binary quadratic form can be written uniquely:??(??, ??) = ??(?? ? ð ??"ð ??"??)(?? ? ð ??"ð ??" ? ??), ?? ? ?, ð ??"ð ??" ? ?.**(19)**

If we let ?? denote the set of such forms, there are commuting actions of ? and ???? 2 (?) on it, and ??/? ? ? as ???? 2 (?) sets. We say that ?? is reduced if|ð ??"ð ??"| > 1 and ? 1 2 ? ?(ð ??"ð ??") < 1 2 , or|ð ??"ð ??"| = 1 and ? 1 2 ? ?(ð ??"ð ??") < 0**(20)**

More explicitly, ??(??, ??) = ???? 2 + ?????? + ???? 2 is reduced if and only if either [9].??? < ?? ? ?? < ?? or 0 ? b ? a = c .

20 Applications of the classification of arithmetic groups

Consequences of the classification of ??-forms. Suppose ?? is an arithmetic subgroup of ????(??, ??), and ?? + ?? ? 5 is odd. Then there is a finite extension ?? of ??, with ring of integers ??, such that ??is commensurable to ????(??, ??), for some invertible, symmetric matrix ?? in Mat ?? ×?? (??). So ?? = ????(??, ??). Restriction of scalars implies there isa group ?? ? that is defined over an algebraic number field ??and has a simple factor that is is ogenous to ??, such that ?? is commensurable to ?? ? ?? . By inspection, we see that a group of the form ????(??, ??) never appears at two places. However, we know that ?? + ?? is odd, so the only possibility for ?? ? ?? is ????(??, ??). Therefore, ?? is commensurable to ????(??, ??) [5,1,12].

21 VII. Conclusion

To state the conclusion in our applications, we can expand binomial and obtain an equation. The coefficient group ?? is sometimesnon-abelian. In case, ?? 1 (??, ??) is a set with no obvious algebraic structure. However, if ?? is an abelian group (as is often assumed ingroup cohomology), ?? 1 (X, M) is an abelian group.The general principle: if ??is an algebraic object that is defined over ?, then ? 1 Gal(?/?), Aut(?? ?) is in one-to-one correspondence with the set of ?-isomorphism classes of ?-defined objects whose ?points are isomorphic to ?? ? . We will explain how to find all of the possible ?-forms of ????(??, ?). The techniques can be used algebraic structure, but additional calculations are needed.

Figure 1:

Figure 2:

Figure 3:

Figure 4:

Figure 5:

only if ?? ?? ? ?.

Figure 6:

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